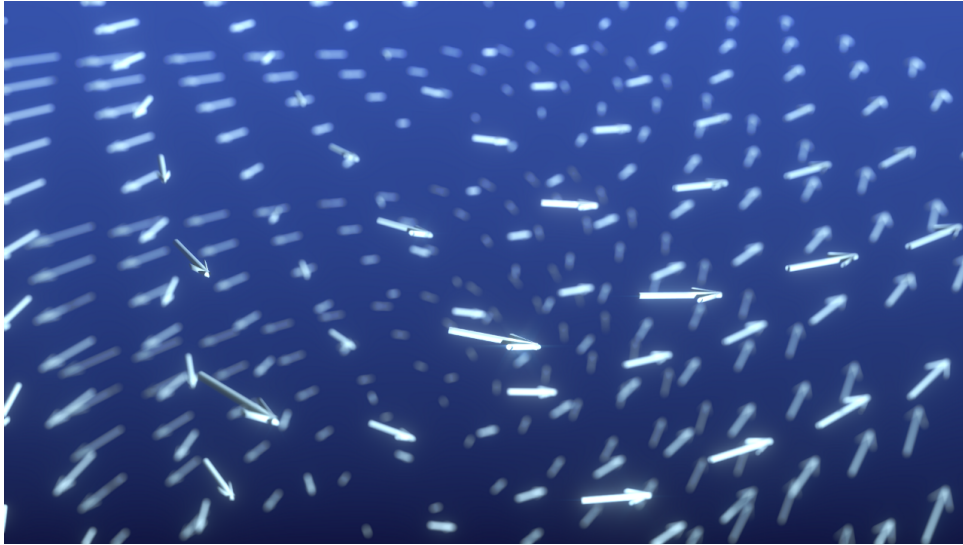# Mathematical warm-up



One of the principal objects of research in my department of knowledge is to find the point of view from which the subject appears in the greatest simplicity.

Josiah Willard Gibbs[1]

Welcome to the first problem set in FYS1120 this year. We're starting out with some exercises on vector calculus and following up with the relevant theorems you need to know this semester. Before you get started, we suggest you have a look at the notes on vector calculus that we've put up on our web pages:

http://elektromagnetisme.no/notes/

The first exercises are basic vector addition, subtraction and products. This might seem trivial to some of you, so you may skip ahead to exercise 0.2 if you'd like, but remember that some repetition never hurts.

## Exercise 0.1: Basic vector calculus

Two vectors are defined as $\mathbf{v} = 1\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $u = -3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}}$. This is the equivalent to $\mathbf{v} = [1, 4, -2]$ and $\mathbf{u} = [-3, 2, 8]$, but in this course we're mostly using the first type of notation.[2]

a) Calculate $\mathbf{v} + 2\mathbf{u}$.

*Answer:* $-5\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 14\hat{\mathbf{k}}$

b) Calculate $3\mathbf{u} - 2\mathbf{v}$

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[1] The quote is from A L Mackay, Dictionary of Scientific Quotations (London, 1994), via Wikiquote

[2] If you'd like to know more about the notation used in this course, check out our page on notation: http://elektromagnetisme.no/tips-and-tricks/

*Answer:* $-11\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 28\hat{\mathbf{k}}$

c) Calculate $\mathbf{v} \cdot \mathbf{u}$.

*Answer:* $-11$

d) Calculate $\mathbf{w} = \mathbf{v} \times \mathbf{u}$.

*Answer:* $36\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 14\hat{\mathbf{k}}$

e) In what direction does $\mathbf{w}$ point relative to the two vectors $\mathbf{v}$ and $\mathbf{u}$?

*Answer:* $\mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{u}$.

In this course we'll for the most part be working with arbitrary vectors (without a specified position and length). This is very powerful both in analytical and numerical solutions of electromagnetism, because you're then able to solve problems for *any* vector on a specified form rather than just a couple of few. In the following we'll work with the arbitrary vectors $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ and $\mathbf{u} = d\hat{\mathbf{i}} + e\hat{\mathbf{j}} + f\hat{\mathbf{k}}$.

## Exercise 0.2: Flux and Divergence

Divergence is a measure of the *expansion* of the vector field and it's denoted $\boldsymbol{\nabla} \cdot \mathbf{F}$. Let's first get some practice calculating the divergence. Find $\boldsymbol{\nabla} \cdot \mathbf{F}$ of the following fields.

a) $\mathbf{F} = x\hat{\mathbf{i}}$

*Answer:*
$$\boldsymbol{\nabla} \cdot \mathbf{F} = 1$$

b) $\mathbf{F} = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

*Answer:*
$$\boldsymbol{\nabla} \cdot \mathbf{F} = 3$$

c) $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{G}$. That is, find the divergence of a field which is itself the curl of a field $\mathbf{G}$.

*Hint: Start from the definition of the curl of* $\mathbf{G}$.

Check your results on the fields in (a) and (b) with your intuition[3]. You might want to plot or make a rough sketch of the fields. Two dimensions will do.

The vector field from a point charge or a sphere of charge centered at the origin have the form $\mathbf{F} = \hat{\mathbf{r}}/r^2$ (see figure 1). In Cartesian coordinates $r^2 = x^2 + y^2 + z^2$ and

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[3]Ask yourself weather a tiny sphere centered at the point in question would have a positive or negative flux.
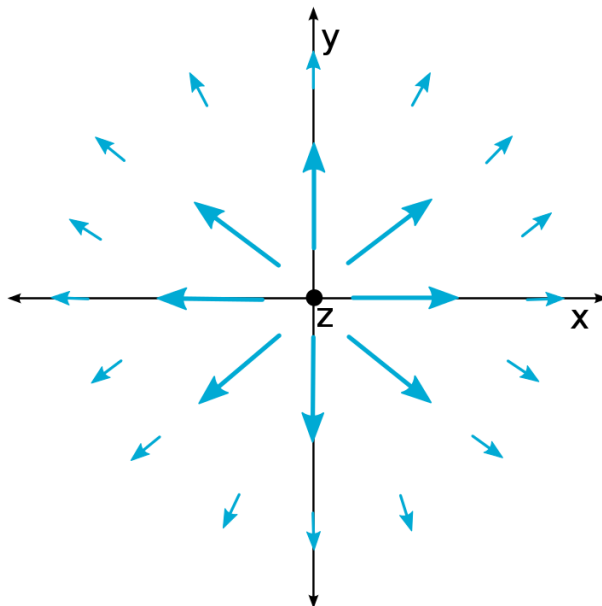
Figure 1: Field from a point charge or a sphere of charge centered at the origin

$$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{\sqrt{x^2 + y^2 + z^2}},$$

such that

$$\mathbf{F} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{3/2}}.$$

d) Calculate the flux of $\mathbf{F}$ trough a sphere of radius $r$ centered at the origin.

*Answer:*

$$\oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{A} = 4\pi$$

e) Show that the divergence of $\mathbf{F}$ is zero everywhere except at the origin.

f) Now use this result and the divergence theorem to calculate the flux trough any closed surface which does not enclose the origin.

## Exercise 0.3: Line integrals and Curl

You've now gotten some practice on calculating the divergence of a vector field, $\boldsymbol{\nabla} \cdot \mathbf{F}$ which returns a *scalar function*. The curl of a vector field, $\boldsymbol{\nabla} \times \mathbf{F}$ is itself a vector field. This is a measure of the *rotation* in the field. Find $\boldsymbol{\nabla} \times \mathbf{F}$ of the following fields.

a) $\mathbf{F} = y\hat{\mathbf{i}}$

*Answer:*

$$\boldsymbol{\nabla} \times \mathbf{F} = -\hat{\mathbf{k}}$$

b) $\mathbf{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$

   *Answer:*
$$\nabla \times \mathbf{F} = -2\hat{\mathbf{k}}$$

c) $\mathbf{F} = \nabla V$. That is, find the curl of a field which is the gradient of a potential $V$.

   *Hint: Start from the definition of the gradient of $V$.*

   *Answer:*
$$\nabla \times \mathbf{F} = 0$$

Check your results on the fields in (a) and (b) with your intuition[4]. You might want to plot or make a rough sketch of the fields. Again two dimensions will do.

An important field in magnetism has the form $\mathbf{F} = \hat{\boldsymbol{\phi}}/r$ where $r$ is the distance away from the z-axis and $\hat{\boldsymbol{\phi}}$ is a unit vector pointing in the direction of the azimuthal angle. In Cartesian coordinates $r = \sqrt{x^2 + y^2}$ and
$$\hat{\boldsymbol{\phi}} = \frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{\sqrt{x^2 + y^2}},$$
such that
$$\mathbf{F} = \frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{x^2 + y^2}.$$

d) Find $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ for a circle of radius $r$ parallel with the $xy$-plane and centered at the z-axis.

   *Answer:*
$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 2\pi$$

e) Show that the curl of $\mathbf{F}$ is zero everywhere except at the z-axis.

f) Use the result from (e) and Stokes' theorem to find $\oint \mathbf{F} \cdot d\mathbf{r}$ for any closed loop which is does not circulate the $z$-axis.

## Useful constants

These constants might be useful in some of this week's exercises.

| | |
|---|---|
| Electrical permittivity in vacuum: | $\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2N^{-1}m^{-2}}$ |
| Gravitational constant: | $G = 6.67 \times 10^{-11}\,\mathrm{m^2kg^{-1}s^{-2}}$ |
| Proton mass: | $m_p = 1.67 \times 10^{-27}\,\mathrm{kg}$ |
| Proton charge: | $q_p = 1e = 1.602 \times 10^{-19}\,\mathrm{C}$ |
| Electron mass: | $m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$ |
| Electron charge: | $q_e = -1e = -1.602 \times 10^{-19}\,\mathrm{C}$ |

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[4]Ask yourself whether and which way a tiny paddle wheel centered at the point in question would rotate.